

## Nuclear Vibrations, Rotations, and the Shell Model. II\*

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Within the  $s$ - $d$  shell of the nucleus it is possible to approximate an interaction between particles having any given range by a sum of an extreme short-range interaction and an extreme long-range interaction. The long-range interaction is nearly equal to the  $P_2$  interaction. This approximation is applied to the vibrational theory and it is shown that rotational spectra are predominant for all ranges of interaction.

THE vibrational theory for nuclei which has been presented by Flamm, Meshkov, and Levinson<sup>1</sup> (this paper will be referred to as I) is utilized in order to investigate the dependence of the collective vibrations on the range of the two-body residual interaction. It is shown that, within the  $s$ - $d$  shell, reasonable interactions can be approximated, for any given range, by a linear combination of two interactions, an extreme long-range interaction, and an extreme short-range interaction. The long-range interaction is found to be nearly equal to the  $P_2$  interaction. This decomposition of the residual interaction is used to investigate the simple problem of vibrations for the case of two particles in the  $s$ - $d$  shell. It is shown that the long-range part of the force tends to be dominant, which produces a rotational spectrum and a strongly deformed nucleus. It is shown that for long-range forces the Elliot<sup>2</sup> wave functions form a good representation for the system.

### APPROXIMATION OF THE INTERACTION

A scalar two-particle operator, which is symmetric with respect to the exchange of particles, has only ten nonvanishing matrix elements in the nuclear  $s$ - $d$  shell.

$$\begin{aligned}
 G_1 &= \frac{1}{5} \langle d^2 0 | V | d^2 0 \rangle, \\
 G_2 &= (1/7) \langle d^2 2 | V | d^2 2 \rangle, \\
 G_3 &= (3/35) \langle d^2 4 | V | d^2 4 \rangle, \\
 G_4 &= (1/14^{1/2}) \langle d^2 2 | V | sd 2 \rangle_s, \\
 G_5 &= (1/5^{1/2}) \langle s^2 0 | V | d^2 0 \rangle, \\
 G_6 &= {}_s \langle sd 2 | V | sd 2 \rangle_s, \\
 G_7 &= \langle s^2 0 | V | s^2 0 \rangle, \\
 G_8 &= \langle d^2 1 | V | d^2 1 \rangle, \\
 G_9 &= {}_a \langle sd 2 | V | sd 2 \rangle_a, \\
 G_{10} &= \langle d^2 3 | V | d^2 3 \rangle.
 \end{aligned} \tag{1}$$

The wave functions are denoted by  $|l_1 l_2 L\rangle$ , where  $L$  is the total angular momentum. The index  $s$  or  $a$  indicates whether the function is symmetric or anti-

symmetric with respect to an exchange of the particles. The matrix elements  $G_1$  to  $G_7$  contain functions symmetric under the exchange of two particles whereas  $G_8$  to  $G_{10}$  contain antisymmetric functions. In order to calculate these matrix elements, it is convenient<sup>3</sup> to expand the interaction between two particles in terms of Laguerre polynomials which constitute a complete set of functions.

The Laguerre polynomials are defined by

$$L_n^{(1/2)} = \sum_k \binom{n+\frac{1}{2}}{n-k} (-1)^k \frac{\rho^k}{k!}, \tag{2}$$

where

$$\rho = |r_{12}|^2 / 2b^2; \tag{3}$$

$b$  is the range parameter of the harmonic oscillator potential, i.e.,

$$b = (\hbar/m\omega)^{1/2}. \tag{4}$$

The Laguerre polynomials satisfy the orthogonality relation

$$\int_0^\infty L_m^{(1/2)}(\rho) L_n^{(1/2)}(\rho) e^{-\rho} \rho^{1/2} d\rho = \delta_{mn} \left(\frac{\pi}{2}\right)^{1/2} \binom{n+\frac{1}{2}}{n}. \tag{5}$$

The interaction between particles can be expanded as

$$V(|r_{12}|) = \sum_n J_n L_n^{(1/2)}(\rho). \tag{6}$$

The  $J_n$  are obtained by using Eq. (5).

Thus,

$$J_n = \frac{2}{\sqrt{\pi}} \binom{n+\frac{1}{2}}{n}^{-1} \int_0^\infty L_n^{(1/2)}(\rho) V(\rho) e^{-\rho} \rho^{1/2} d\rho. \tag{7}$$

By using the definition for  $L_n^{(1/2)}$  and the binomial coefficient property

$$\binom{n}{k} \frac{2^{k+1}}{(2k+1)!!} = \frac{2}{k!} \binom{n+\frac{1}{2}}{n-k} \binom{n+\frac{1}{2}}{n}^{-1}, \tag{8}$$

where

$$(2k+1)!! = 1 \times 3 \times 5 \times \dots \times (2k+1),$$

$J_n$  is obtained in the form

$$J_n = \sum_k \frac{(-1)^k}{\sqrt{\pi}} \binom{n}{k} \frac{2^{k+1}}{(2k+1)!!} \int_0^\infty e^{-\rho} \rho^{n+\frac{1}{2}} V(\rho) d\rho. \tag{9}$$

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<sup>1</sup> E. Flamm, S. Meshkov, and C. A. Levinson, preceding paper [Phys. Rev. **129**, 297 (1963)].

<sup>2</sup> J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128 and 562 (1958). M. K. Bannerjee and C. A. Levinson (to be published).

<sup>3</sup> J. P. Elliott, lectures given at Princeton University, 1958 (unpublished).

It is possible to relate the  $J_n$  integrals to the Talmi  $I_n$  integrals defined as

$$I_n = \frac{2^{n+1}}{(2n+1)!!} \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\rho} \rho^{n+\frac{1}{2}} V(\rho) d\rho. \quad (10)$$

Thus,

$$J_n = \sum_k (-1)^k \binom{n}{k} I_k. \quad (11)$$

The determination of the matrix elements of the polynomials  $L_n^{(1/2)}$  is straightforward. For the  $s$ - $d$  shell it is noted that polynomials with  $n \leq 4$  have nonvanishing matrix elements.

The polynomials with nonvanishing matrix elements in the  $s$ - $d$  shell are

$$\begin{aligned} L_0^{(1/2)} &= 1, \\ L_1^{(1/2)} &= \frac{3}{2} \left(1 - \frac{2}{3}\rho\right), \\ L_2^{(1/2)} &= \frac{15}{8} \left(1 - \frac{4}{3}\rho + \frac{4}{15}\rho^2\right), \\ L_3^{(1/2)} &= \frac{35}{16} \left(1 - 2\rho + \frac{4}{5}\rho^2 - \frac{8}{105}\rho^3\right), \\ L_4^{(1/2)} &= \frac{315}{128} \left(1 - \frac{8}{3}\rho + \frac{8}{5}\rho^2 - \frac{32}{105}\rho^3 + \frac{16}{945}\rho^4\right). \end{aligned} \quad (12)$$

Since the polynomial  $L_0^{(1/2)}$  is a constant, it does not contribute to level splitting. Within a major shell of the same parity, the polynomial  $L_1^{(1/2)}$  also does not contribute to the splitting. This is seen by writing  $L_1^{(1/2)}$  in terms of  $r_{12}$  (the distance between particles).

$$L_1^{(1/2)} = \frac{3}{2} \left[ 1 - \frac{1}{3b^2} (r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta) \right]. \quad (13)$$

The matrix elements of  $r_i^2$  are a constant within a major shell. The term containing  $\cos\theta$  has a zero matrix element because it has negative parity.

Thus, we have shown that in the  $s$ - $d$  shell only  $L_2^{(1/2)}$ ,  $L_3^{(1/2)}$ , and  $L_4^{(1/2)}$  contribute to the level splitting. Therefore, the level splitting of an arbitrary potential  $V$  is determined by three parameters  $J_2$ ,  $J_3$ , and  $J_4$ .

The  $J$ 's can be determined by using the known Talmi integrals and Eq. (11). The Talmi integrals for various potential shapes are given by Thieberger.<sup>4</sup>

For the case of a Gaussian potential,

$$V(r_{12}) = V_0 \exp(-r_{12}/r_0)^2, \quad (14)$$

the Talmi integrals are

$$I_n = V_0 \eta^{\frac{3}{2}+n},$$

where  $\eta$  is the dimensionless parameter

$$\eta = r_0^2 / (r_0^2 + 2b^2). \quad (15)$$

<sup>4</sup>I. Talmi, *Helv. Phys. Acta* **25**, 185 (1952); R. Thieberger, *Nucl. Phys.* **2**, 533 (1956-7).

By using this result, we find that

$$J_n = V_0 \eta^{3/2} (1-\eta)^n. \quad (16)$$

The parameter  $\eta$  varies from zero to one as the range of the Gaussian potential  $r_0$  from zero to infinity.

For extreme short ranges of the potential, the factor  $\eta^{3/2}$  may be absorbed in the constant  $V_0$ .

A  $\delta$ -type potential is obtained by letting  $\eta$  approach zero and keeping

$$\lim_{\eta \rightarrow 0} V_0 \eta^{3/2} = 1.$$

In this limit

$$J_n = 1. \quad (17)$$

In the extreme long-range limit of the potential

$$r_0 \gg b, \quad \text{and, therefore, } \eta \sim 1.$$

For this reason, we have

$$J_0 \gg J_1 \gg J_2 \gg J_3 \gg J_4.$$

Since  $L_0^{(1/2)}$  and  $L_1^{(1/2)}$  do not contribute to the splitting and since  $J_3$  and  $J_4$  are much smaller than  $J_2$ , we find that the level splitting is essentially determined by  $L_2^{(1/2)}$  only.

The  $L_2^{(1/2)}$  polynomial contains some nonsplitting terms, specifically the constant and the first power of  $\rho$ . Thus, the only significant splitting term is

$$\begin{aligned} \frac{1}{8b^2} r_{12}^4 &= \frac{1}{8b^2} (r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta)^2 \\ &= \frac{1}{8b^2} \left[ r_1^4 + r_2^4 + \frac{8}{3} r_1^2 r_2^2 \frac{3 \cos^2\theta - 1}{2} \right. \\ &\quad \left. + \frac{4}{3} r_1^2 r_2^2 + 2r_1^2 r_2^2 - 4r_1 r_2 \cos\theta (r_1^2 + r_2^2) \right]. \end{aligned} \quad (18)$$

Since within a major shell

$$\langle r_1^4 \rangle = \frac{3}{8} [(2N+3)^2 + 1] - \frac{1}{2} l(l+1), \quad (19)$$

and since

$$(3 \cos^2\theta - 1)/2 = P_2(\cos\theta), \quad (20)$$

we obtain

$$\begin{aligned} L_2^{(1/2)} &= \frac{1}{8b^2} \left[ -\frac{1}{2} (\mathbf{1}_1^2 + \mathbf{1}_2^2) + \frac{8}{3} r_1^2 r_2^2 P_2(\cos\theta) \right. \\ &\quad \left. + \text{nonsplitting terms} \right]. \end{aligned} \quad (21)$$

This shows that in the long-range limit any reasonable potential contains a considerable amount of the so-called  $P_2$  potential,  $V = r_1^2 r_2^2 P_2(\cos\theta)$ .

The matrix elements of the  $L_2^{(1/2)}$ ,  $L_3^{(1/2)}$ , and  $L_4^{(1/2)}$  polynomials are given in Table I. It is observed that the symmetric matrix elements of  $L_3^{(1/2)}$  and  $L_4^{(1/2)}$  have a nearly constant ratio. This means that the parameters  $J_3$  and  $J_4$  do not enter into the various matrix elements independently, but rather as a linear

TABLE I. Matrix elements of the Laguerre polynomials.

	$L_0^{(1/2)}$	$L_1^{(1/2)}$	$L_2^{(1/2)}$	$L_3^{(1/2)}$	$L_4^{(1/2)}$
$G_1$	1/5	-2/5	2/3	-7/15	21/80
$G_2$	1/7	-2/7	23/84	-11/84	3/56
$G_3$	3/35	-6/35	3/14	-9/70	9/280
$G_4$	0	0	1/3	-11/24	3/16
$G_5$	0	0	2/3	-7/6	21/32
$G_6$	1	-2	23/12	-11/12	3/8
$G_7$	1	-2	35/12	-35/12	105/64
$G_8$	1	-2	11/4	-7/4	0
$G_9$	1	-2	3/2	-1/2	0
$G_{10}$	1	-2	15/8	-7/8	0

combination  $J_3 - \alpha J_4$ . Thus, a potential of any range can be described within the  $s$ - $d$  shell by means of two arbitrary parameters or as a linear combination of two potentials. For convenience, we take these two potentials to be the extreme short-range potential  $V_s$ , defined as

$$V_s = L_0^{(1/2)} + L_1^{(1/2)} + L_2^{(1/2)} + L_3^{(1/2)} + L_4^{(1/2)}, \quad (22)$$

and the extreme long-range potential  $V_L$  defined as

$$V_L = \frac{1}{2} L_2^{(1/2)}. \quad (23)$$

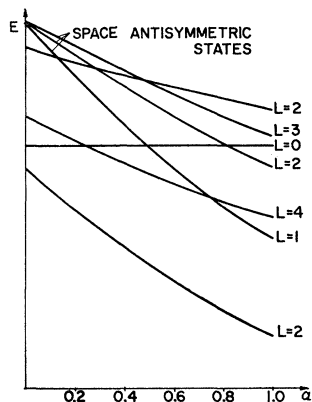


FIG. 1. Energy levels of two particles in the  $s$ - $d$  shell as a function of the range parameter  $a$ .

The matrix elements of these two potentials are given in Table II. We now diagonalize the matrix of the linear combinations of these potentials,

$$V = aV_L + (1-a)V_s, \quad (24)$$

for the simplest case of two particles in the  $s$ - $d$  shell.

TABLE II. Matrix elements of  $V_s$  and  $V_L$ .

	$V_s$	$V_L$
$G_1$	21/80	1/3
$G_2$	3/56	23/168
$G_3$	9/280	3/28
$G_4$	1/16	1/6
$G_5$	5/32	1/3
$G_6$	3/8	23/24
$G_7$	41/64	35/24
$G_8$	0	11/8
$G_9$	0	3/4
$G_{10}$	0	15/16

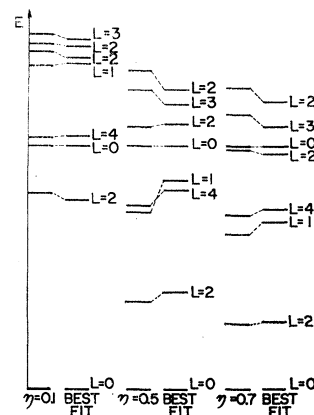


FIG. 2. Approximation of energy levels for a Gaussian interaction for various values of the range parameter  $\eta$  by  $a: aV_L + (1-a)V_s$  interaction.

This diagonalization only involves diagonalizing  $2 \times 2$  matrices. The results of this diagonalization are given in Fig. 1. The over-all magnitude of the potential has been adjusted so that the splitting between the two  $L=0$  levels remains constant. In order to check the accuracy of this approximation, we compare the energy splittings produced by a Gaussian potential for various values of  $\eta$  with those splittings using a variable range in our approximation. We have used a least-squares fit to find the best  $a$  for each  $\eta$ . As shown in Fig. 2, the level splittings are in reasonable agreement. The least-squares fit for  $a$  vs  $\eta$  is given in Fig. 3.

We have shown that within the  $s$ - $d$  shell an arbitrary (range) potential can be approximated fairly well by a sum of two potentials. The short-range part of the potential possesses the characteristics of a pairing potential, i.e., it depresses one energy very deeply compared to the other energy levels. The long-range part of the potential is nearly of the  $P_2$  type and will produce a rotational spectrum, as is well known.

APPLICATION TO VIBRATIONS

The formalism developed in I and the variable range approximation developed in the previous section will be used in order to investigate the collective behavior for the simple case of two particles in the  $s$ - $d$  shell. This case is investigated since the vibrational theory of I is exact for this case, which allows a clear determination of the validity of the approximations that will be used.

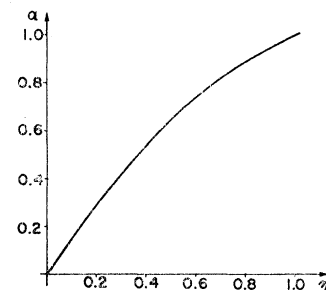
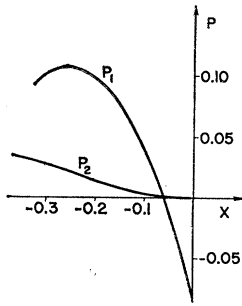
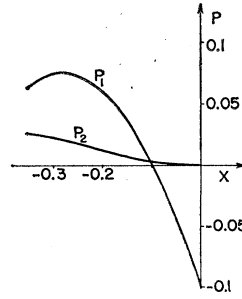


FIG. 3. Dependence of the range parameter  $a$  on the range parameter  $\eta$  of the Gaussian interaction.

FIG. 4.  $P_1$  and  $P_2$  as a function of  $x$  for  $a=1.0$ .FIG. 6.  $P_1$  and  $P_2$  as a function of  $x$  for  $a=0.4$ .

The intrinsic wave function we will use is defined by

$$\Phi(x,12) = [-(3x+1)S_0(1)+d_0(1)] \times [-(3x+1)S_0(2)+d_0(2)]. \quad (25)$$

For  $x=0$  this function is equal to the lowest Elliot wave function except for a normalization factor. The function  $\Phi(x,12)$  is equivalent to the function used in I but for the vibrational parameter  $x$ , which is defined differently.

By using this wave function, it is possible to expand  $H\Phi(x,12)$  in terms of five independent functions,

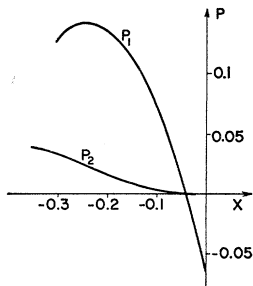
$$\Phi, \mathbf{L}^2\Phi, \mathbf{L}^4\Phi, \frac{d}{dx}\Phi, \text{ and } \frac{d^2}{dx^2}\Phi.$$

This expansion may be written as

$$H\Phi(x,12) = - \left[ P_{0a}(x) + P_{0b}(x)L^2 + P_{0c}(x)L^4 + P_1(x)\frac{d}{dx} + P_2(x)\frac{d^2}{dx^2} \right] \Phi(x,12). \quad (26)$$

The  $P$  polynomials are calculated as in I and are given by

$$P_2 = -\frac{1}{9} \left[ \left( -\frac{5}{2}G_1 + 7G_2 + 5G_4 - \frac{1}{4}G_5 - G_6 + \frac{1}{2}G_7 \right) + x(-15G_1 + 42G_2 + 57G_4 - 15G_5 - 6G_6 + 3G_7) + x^2 \left( -\frac{45}{2}G_1 + 63G_2 + 189G_4 - \frac{135}{2}G_5 - 9G_6 + \frac{9}{2}G_7 \right) + x^3(189G_4 - 135G_5) - x^4\frac{405}{4}G_5 \right], \quad (27a)$$

FIG. 5.  $P_1$  and  $P_2$  as a function of  $x$  for  $a=0.7$ .

$$P_1 = -\left[ \frac{1}{3}(-7G_2 - 5G_4 + G_6) + (-7G_2 - 14G_4 + G_6)x - 21G_4x^2 \right], \quad (27b)$$

$$P_{0a} = - \left[ \left( 5G_1 + \frac{15}{6}G_5 \right) + 15G_5x + \frac{45}{2}G_5x^2 \right], \quad (27c)$$

$$P_{0b} = - \left[ \frac{1}{12} \left( -13G_1 + 20G_2 - 3G_3 + 20G_4 - \frac{13}{2}G_5 \right) + \left( 5G_4 - \frac{13}{4}G_5 \right)x - \frac{39}{8}G_5x^2 \right], \quad (27d)$$

$$P_{0c} = -\frac{1}{24} \left[ (G_1 - 2G_2 + G_3 - 2G_4 + \frac{1}{2}G_5) + (-6G_4 + 3G_5)x + \frac{9}{2}G_5x^2 \right]. \quad (27e)$$

The case of "freezing" is defined by the condition that there exists a value  $x=x_0$ , for a given Hamiltonian such that

$$P_1(x_0) = P_2(x_0) = 0. \quad (28)$$

In this case it is possible to operate on Eq. (26) with the projection operator  $P^L$ . This operator projects out that part of  $\Phi$  which has an angular momentum  $L$ . Thus, since  $P^L$  commutes with the Hamiltonian,

$$[P^L, H] = 0, \quad (29)$$

it is evident that

$$HP^L\Phi(x_0,12) = - \left[ P_{0a}(x_0) + P_{0b}(x_0)L(L+1) + P_{0c}(x_0)L^2(L+1)^2 \right] P^L\Phi(x_0,12). \quad (30)$$

Therefore,  $P^L\Phi(x_0,12)$  is an eigenfunction of the Hamiltonian with the eigenvalue

$$E_L = -P_{0a}(x_0) - P_{0b}(x_0)L(L+1) - P_{0c}(x_0)L^2(L+1)^2. \quad (31)$$

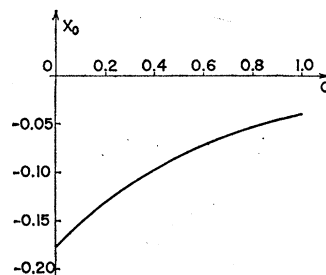
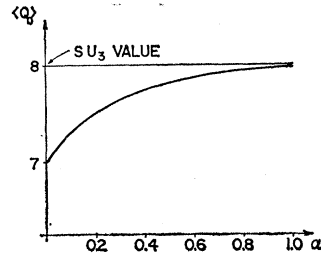
FIG. 7. The freezing point  $x_0$  as a function of the range parameter  $a$ .

FIG. 8. The quadrupole deformation  $\langle Q_0 \rangle$  as a function of the range parameter  $a$ .  $\langle Q_0 \rangle = 8$  is the deformation for the Elliott wave functions based on the group  $SU_3$ .



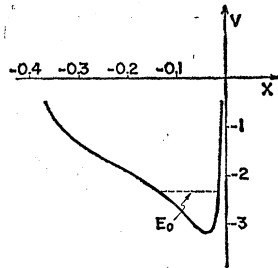
Physically the existence of the eigenfunction  $P^L\Phi(x_0, 12)$  that contains only one value of  $x$ , namely,  $x_0$ , means that there are actually no vibrations taking place and that the nucleus only rotates. In this case the nucleus shall be said to be "frozen at the point  $x_0$ ."

The energy spectrum for the case of freezing is almost that of a rigid rotator. The term containing  $L^2(L+1)^2$  is a small correction for all cases of physical interest.

We shall show that the freezing condition in Eq. (28) is nearly satisfied for reasonable interactions. The polynomials  $P_1$  and  $P_2$  are plotted as a function of  $x$  for various values of the range parameter of the interaction  $a$ , Figs. 4 to 7. The nucleus is nearly frozen because there exists a point  $x_0$  for which  $P_1$  and  $P_2$  are both very small. The freezing condition is satisfied better for long-range interactions than for short-range interactions. In order to check the accuracy of the "freezing" approximation, the approximate eigenfunction  $P^L\Phi(x_0, 12)$  is compared with the exact eigenfunction obtained by diagonalizing the interaction Hamiltonian for various values of the range parameter  $a$ . The overlap between these two functions is calculated for  $L=0$  and  $L=2$ . For long-range interactions this overlap is nearly 100%. As the range becomes shorter the overlap gets smaller. However, even for the case  $a=0$  (an extreme short-range interaction) the overlap is greater than 98%. Therefore although the freezing approximation for short-range interactions is not as good as for long-range interactions, it is still a very good approximation.

It is also possible to investigate the dependence of the freezing point  $x_0$  on the range of interaction (Fig. 7). The freezing point defined by Eq. (28) is taken to be

FIG. 9. The one-dimensional potential  $V$  as a function of  $x$  for a long-range interaction ( $a=1.0$ ), for  $L=0$ .



that point where  $P_1(x)=0$ , since  $P_1$  changes more rapidly than  $P_2$ . It is seen that for long-range interactions the freezing point is nearly equal to the point  $x=0$  which defines the Elliott function. As the range of the force decreases,  $x_0$  becomes larger. This shows that the Elliott representation is better for long-range interactions than for short-range interactions.

For the case of freezing it is possible to calculate the average of the quadrupole operators defined within a major shell by

$$Q_0 = 2(4\pi/5)^{1/2} (r^2/b^2) Y_0^2(\theta, \varphi), \quad (32)$$

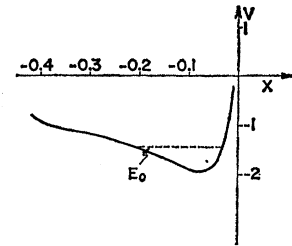
where  $Y_0^2$  is the spherical function. The average deformation is given by

$$\frac{\langle \Phi(x_0, 12) | Q_0(1) + Q_0(2) | \Phi(x_0, 12) \rangle}{\langle \Phi(x_0, 12) | \Phi(x_0, 12) \rangle} = \langle Q_0 \rangle. \quad (33)$$

This deformation can be expressed in terms of  $x_0$

$$\langle Q_0 \rangle = 8(2x_0 + 1) / (3x_0^2 + 2x_0 + 1). \quad (34)$$

FIG. 10. The one-dimensional potential  $V$  as a function of  $x$  for a short-range potential ( $a=0$ ), for  $L=0$ .



In Fig. 8 the dependence of the deformation on the range of interaction is shown. For long ranges of interaction the nucleus is strongly deformed. As the range of interaction becomes smaller, the nucleus is less deformed. However, even for the extreme short-range case, the nucleus is still far from spherical. It should be stressed that this deformation is the deformation of the outer shell only.

A pictorial description of what is meant by saying that the freezing is stronger for long-range forces than for short-range forces is given in Figs. 9 and 10. The one-dimensional potential  $V$ , defined in I, in which the nucleus vibrates is plotted as a function of  $x$ . It is seen that the potential is much deeper and narrower for the long-range interaction (Fig. 9), i.e., the freezing is stronger for this case than for the short-range interaction.

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